Definition

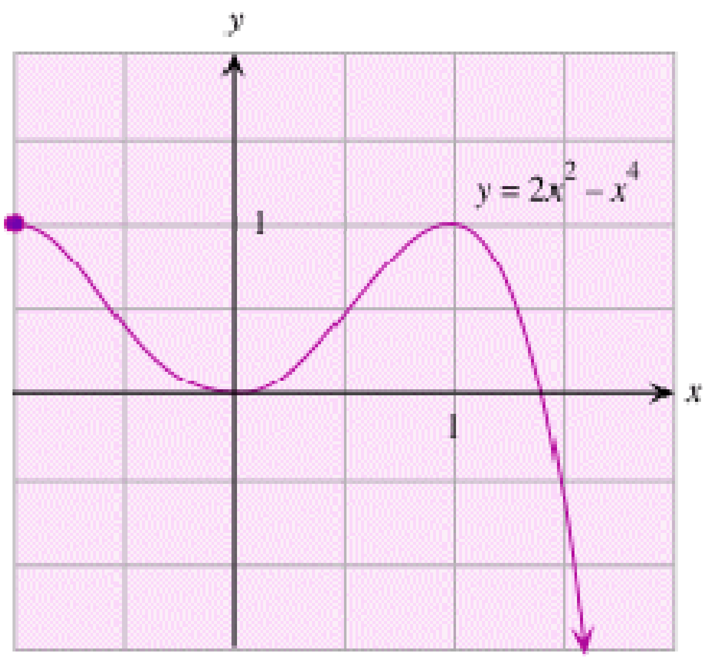
A function f is increasing over I if, for every a and b in I,

$$\text{if } a < b, \text{ then } f(a) < f(b)$$

A function f is decreasing over I if, for every a and b in I,

$$\text{if } a < b, \text{ then } f(a) > f(b).$$

Consider the function f below.



Where is it increasing?

Where is it decreasing?

Sketch some tangent lines to the curve.

Where do the tangent lines have a positive slope?

Where do the tangent lines have a negative slope?

What can we say about the sign [positive/negative] of the derivative f' when the function f is increasing? Decreasing?

For a function in general, we can use the derivative to determine whether a function is increasing or decreasing.

Theorem 1

If $f'(x) > 0$ for all x in an interval I, then *f* is increasing over I.
If $f'(x) < 0$ for all x in an interval I, then *f* is decreasing over I.

Critical points

A critical point of a function is an interior point c of its domain at which the tangent line to the graph at (c, *f*(c)) is horizontal or at which the derivative does not exist. That is, c is a critical point if

$$f'(c) = 0 \text{ or } f'(c) \text{ does not exist.}$$

NOTE: A function can change from increasing to decreasing or from decreasing to increasing only at a critical point. However, there may be critical points for which the function does not change from increasing to decreasing or from decreasing to increasing

Definition

Suppose that f is a function whose value *f*(c) exists at input c in the domain of *f*. Then:

f(c) is a **relative minimum** if there exists an open interval I_1 containing c in the domain such that $f(c) \leq f(x)$, for all x in I_1

and

f(c) is a **relative maximum** if there exists an open interval I_2 containing c in the domain such that $f(c) \geq f(x)$, for all x in I_2

Theorem 2

If a function *f* has a relative extreme value *f*(c) on an open interval, then *c* is a critical point so

$$f'(c) = 0 \text{ or } f'(c) \text{ does not exist.}$$

First Derivative Test for Relative Extrema

For any continuous function *f* that has exactly one critical point c in an open interval (a,b):

(1) *f* has a relative minimum at c if $f'(x)<0$ on (a, c) and $f'(x)>0$ on (c, b). That is, *f* is decreasing to the left of c and increasing to the right of c.

(2) *f* has a relative maximum at c if $f'(x)>0$ on (a, c) and $f'(x)<0$ on (c, b). That is, *f* is increasing to the left of c and decreasing to the right of c.

(3) *f* has neither a relative maximum nor a relative minimum at c if $f'(x)$ has the same sign on (a, c) as on (c, b).

Note:

- The derivative f' is used to find the critical points of *f*.
- The test values, in the intervals defined by the critical points, are substituted into the derivative f'.
- The function values are found using the original function *f*.
- Use the derivative f' to find information about the shape of the graph of *f* (increasing, decreasing, relative extrema).

Using the First DerivativeTest to create and understand graphs and to find relative extrema:

1. Find $f'(x)$.
2. Find the critical point(s) *(candidates for relative extrema)*
3. Partition the number line using the critical points.
4. Pick test numbers for each interval to evaluate in the derivative f' to determine where the derivative is positive or negative. If $f' > 0$ on an interval then the function *f* is ________________. If $f' < 0$ on an interval then the function *f* is ______________ .
5. Apply the First-Derivative Test to determine where the relative extrema occur.
6. Find the coordinates of the relative extrema by substituting the critical numbers into the function *f.*
7. Sketch the graph.
8. Verify your answer(s) with your graphing utility.

Section 2.2: Using the Second Derivative

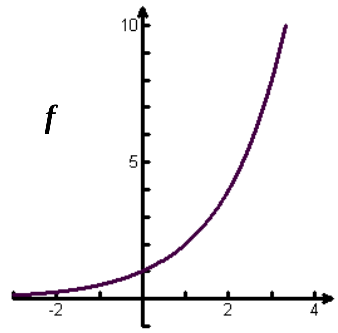
Concave up:

Consider the graph of the functions *f*.

$f(x)$ is ___________________.

$f'(x)$ is ___________________.

$f''(x)$ is ___________________.



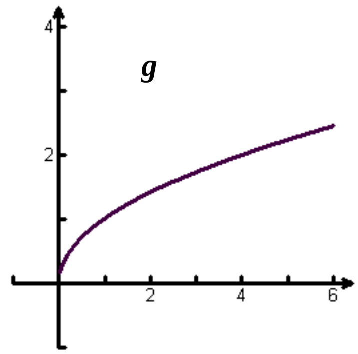
Concave down:

Consider the graph of the functions g.

$g(x)$ is ___________________.

$g'(x)$ is ___________________.

$g''(x)$ is ___________________.

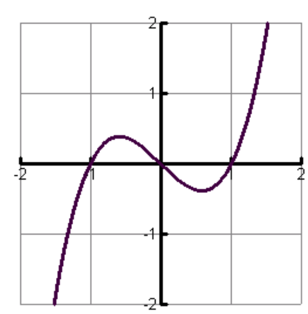


Suppose that *f* is a function whose derivative *f'* exists at every point in an open interval *I*. Then
1. *f* is **concave up** on the interval *I* if *f '* is increasing over *I*.
2. *f* is **concave down** on the interval *I* if *f '* is decreasing over *I*.

Test for Concavity

1. If $f''(x) > 0$ on an interval *I*, then the graph of *f* is turning up. That is, *f '* is increasing, so *f* is **concave up** on *I*.
2. If $f''(x) < 0$ on an interval *I*, then the graph of *f* is turning down. That is, *f '* is decreasing, so *f* is **concave down** on *I*.

NOTE: Concavity and increasing/decreasing are independent concepts but the increasing/decreasing aspects of the derivative tell us about the function's concavity.



$h(x) = x^3 - x$

Example 1:

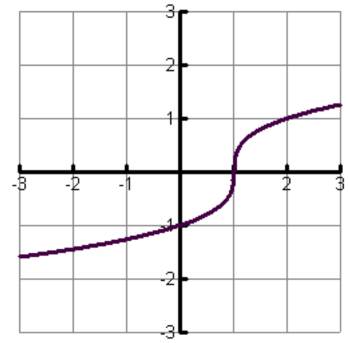
a) Find $h''(x)$.

b) Find $h''(-1)$, $h''(0)$, and $h''(1)$.

c) Give the intervals of concavity.

.**Example 2:**

$k(x) = \sqrt[3]{x-1}$



a) Find $k''(x)$.

b) Find $k''(-1)$, $k''(1)$ and $k''(2)$.

c) Give the intervals of concavity.

Second-Derivative test for Relative Extrema

Suppose that *f* is a function for which *f* ’(*x*) exists for every *x* in an open interval (*a*, *b*) contained in its domain, and that there is a critical point *c* in (*a*, *b*) for which *f* ’(*c*) = 0. Then:

1. *f*(*c*) is a relative minimum if $f''(c) > 0$.
2. *f*(*c*) is a relative maximum if $f''(c) < 0$.

The test fails if $f''(c) = 0$. The First Derivative Test would then have to be used.

An **inflection point is** a point across which the direction of concavity changes.

Finding Points of Inflection

If a function *f* has a point of inflection, it occurs at a point x_0, where

$f''(x_0) = 0$ or $f''(x_0)$ does not exist.

Strategy for Sketching Graphs of *f*

1. Find the first- and second-derivative.
2. Find the critical points of *f*. *(use the first derivative)*
3. Find the relative maxima and minima.
4. Find any inflection points. *(use the second derivative)*
5. Determine the concavity of the function.
6. Sketch the graph.

Section 2.3: Asymptotes & Rational Functions

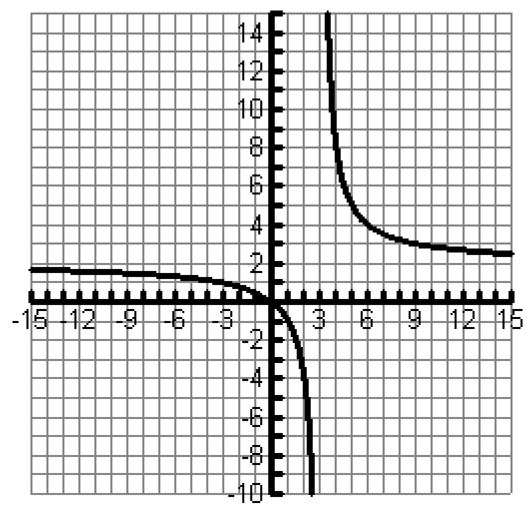
A **rational function** $f(x)$ is the quotient of two polynomial functions, $p(x)$ and $q(x)$ such that $q(x) \neq 0$:

$$f(x) = \frac{p(x)}{q(x)}, q(x) \neq 0$$

The domain of f consists of all real numbers except those for which the denominator q is zero; Domain = $\{x \mid q(x) \neq 0\}$.

Vertical and Horizontal Asymptotes of Rational Functions $f(x) = \frac{p(x)}{q(x)}$

Consider the rational function $f(x) = \frac{2x}{x-3}$



1. What is the domain of f?

2. Complete the chart below.

x	$f(x)$	x	$f(x)$
3.2		2.8	
3.1		2.89	
3.01		2.9	
3.001		2.98	
3.0001		2.99	

3. Evaluate: $\lim_{x \to 3^+} \left(\frac{2x}{x-3} \right) =$ Evaluate: $\lim_{x \to 3^-} \left(\frac{2x}{x-3} \right) =$

The line $x = a$ is a vertical asymptote if any of the following statements are true:

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

The graph of a rational function never crosses a vertical asymptote. If a rational function is in simplified form and an input value a makes the denominator 0, then $x = a$ is a vertical asymptote.

4. *What is the vertical asymptote of f?*

5. *Sketch the vertical asymptote as a broken line,* - - - - -, *on the graph above.*

6. Complete the chart below for the function $f(x)=\dfrac{2x}{x-3}$.

x	*f(x)*	*x*	*f(x)*
10		-10	
100		-100	
1000		-1000	
10000		-10000	

7. Evaluate: $\lim_{x\to\infty}\left(\dfrac{2x}{x-3}\right)=$ Evaluate: $\lim_{x\to-\infty}\left(\dfrac{2x}{x-3}\right)=$

The line $y = b$ is a horizontal asymptote if either or both of the following statements are true:

$$\lim_{x\to-\infty} f(x)=b \text{ or } \lim_{x\to\infty} f(x)=b$$

The graph of a rational function may or may not cross a horizontal asymptote. Horizontal asymptotes occur when the degree of the numerator is less than or equal to the degree of the denominator.

8. *What is the horizontal asymptote of f?*

9. *Sketch the horizontal asymptote as a broken line,* - - - - -, *on the graph of* $f(x)=\dfrac{2x}{x-3}$

- ✷ When the degree of the numerator is less than the degree of the denominator, the *x*-axis (the line $y = 0$) is the horizontal asymptote.
- ✷ When the degree of the numerator is the same as the degree of the denominator, the line $y = a/b$ is a horizontal asymptote, where *a* is the leading coefficient of the numerator and *b* is the leading coefficient of the denominator.
- ✷ When the degree of the numerator is greater than the degree of the denominator, there is *no* horizontal asymptote.

Oblique Asymptotes

The line $y = mx + b$ is an oblique asymptote of the rational function $f(x)=\dfrac{P(x)}{Q(x)}$, if *f(x)* can be expressed as $f(x)=(mx+b)+g(x)$ where *g(x)* approaches 0 as |x| approaches ∞. Oblique asymptotes occur when the degree of the numerator is exactly 1 more than the degree of the denominator. A graph can cross an oblique asymptote. *We use long division to find oblique asymptotes.*

Strategies for Sketching the Graph of a Function, *f*

1. *Intercepts*. Find the x-intercept(s) (solve $f(x)=0$) and the y-intercept (find $f(0)$ of the graph.

   Note: The x-intercepts of the graph of a rational functions are the real zeros of the numerator.
2. *Asymptotes*. Find the vertical, horizontal, and oblique asymptotes.
3. *Derivatives*. Find f' and f''.
4. *Undefined values and critical points of f*. Find the values of *x* where the function *f* is undefined and the critical points of *f*.
5. *Increasing/Decreasing and Relative Extrema*. Use points found ins *Step (4)* to determine the intervals where *f* is increasing/decreasing. Use this information or the second derivative to determine the relative extrema.
6. *Inflection Points*. Determine candidates of inflection points by finding values of *x* where $f''(x)=0$ or $f''(x)$ does not exist. If a function value *f(x)* does not exist, then the function does not have an inflection point at *x*.
7. *Concavity*. Use the values *x*-values from *Step (6)* to partition the number line. Determine the concavity by checking to see where

   $f''(x)>0$ which mean $f'(x)$ is __________________ and *f* is ____________________on the interval *I*

   and

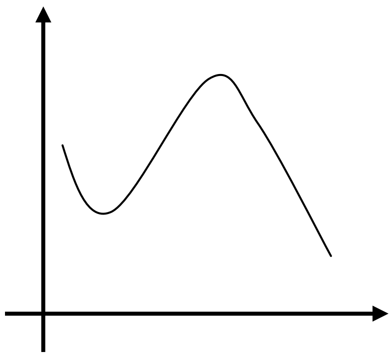
   $f''(x)<0$ which mean $f'(x)$ is __________________ and *f* is ____________________on the interval *I*.

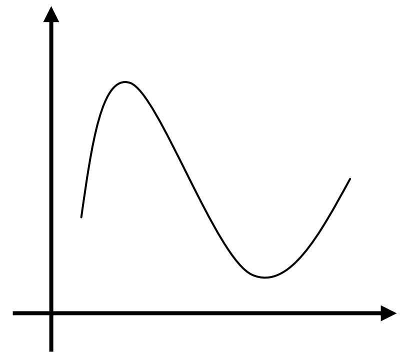
   Do this by selecting test points and substituting into $f''(x)$.
8. *Sketch the graph*. Use the information from *Steps (1-7)* to sketch the graph, plotting extra points [using your calculator] as needed.

Using Derivatives to Find Absolute Maximum & Minimum Values (2.4)

Let *f* be a function whose value *f(c)* exists at input *c* in an interval *I* in the domain of *f*. Then:

- *f(c)* is an **absolute minimum** if $f(c) \le f(x)$ for all *x* in *I*.
- *f(c)* is an **absolute maximum** if $f(c) \ge f(x)$ for all *x* in *I*.





The Extreme Value Theorem: A continuous function *f* defined over a closed interval [a, b] must have an absolute minimum value and an absolute maximum value at points in [a, b].

Maximum–Minimum Principle 1 (Theorem 8)

Suppose that *f* is a continuous function over a closed interval [a, b]. To find the absolute maximum and minimum values of the function over [a, b]:

a) Find $f'(x)$
b) Determine the critical points in [a, b].
c) List the critical points of *f* and the endpoints of the interval: $a, c_1, c_2, ..., c_n, b$.
d) Find the function values in part c): $f(a), f(c_1), f(c_2),...,f(c_n), f(b)$. The largest of these is the **absolute maximum** of *f* over [a,b]. The smallest of these is the **absolute minimum** of *f* over $[a,b]$.

Maximum-Minimum Principle 2 (Theorem 9)

Let *f* be a function such that $f'(x)$ exists for every *x* in an interval *I*, and that there is *exactly one* critical point *c*, interior to *I*, for which $f'(c) = 0$. Then

f(c) is the absolute maximum value over *I* if $f''(c) < 0$*;* or

f(c) is the absolute minimum value over *I* if $f''(c) > 0$.

Strategy for Finding Absolute Maximum and Minimum Values

1. Find $f'(x)$
2. Find the critical points
3. If the interval is closed [ ] and there is more than one critical point, use Max-Min Principle 1.
4. If the interval is closed and there is exactly one critical point, use either Max-Min Principle 1 or 2. If the function is relatively easy to differentiate use Max-Min Principle 2.
5. If the interval is not closed, does not have endpoints, or does not contain its endpoints, such as (-∞, ∞), (0, ∞), or (a, b), and the function has only one critical point, use Max-Min Principle 2. If an interval is not specified then use the domain of that function.

Solving Maximum–Minimum Problems Using Calculus (2.5)

A Strategy for Solving Maximum-Minimum Problems

1. Read the problem carefully. If relevant, make a drawing.
2. Label the picture with appropriate variables and constants, noting what varies and what stays fixed.
3. Translate problem to an equation involving a quantity Q to be maximized or minimized. Try to represent Q in terms of the variables of step (2).
4. Try to express Q as a function of one variable. Determine the maximum or minimum values and the points they occur using the First Derivative Test, Second Derivative Test, Maximum-Minimum Principle 1, or Maximum-Minimum Principle 2.