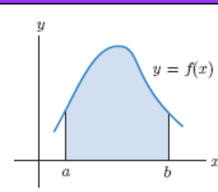
4.1 The Area under a Graph

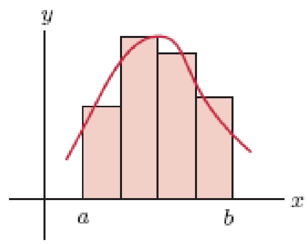
OBJECTIVES

- Use the area under a graph to find total cost.
- Use rectangles to approximate the area under a graph.

Area Under a Curve/Graph

Definition	Example
Area Under the Graph of $f(x)$ *from a to b*: An example of this is shown to the right	

In this section we will learn to estimate the area under the graph of $f(x)$ from $x = a$ to $x = b$ by dividing up the interval into partitions (or subintervals), each one having width $\Delta x = \frac{b-a}{n}$ where n = the number of partitions that will be constructed. In the example below, $n = 4$.



4.1 The Area Under a Graph

Example 1: Covington Bakers determines that the cost, in cents per pound, of its sourdough bread is

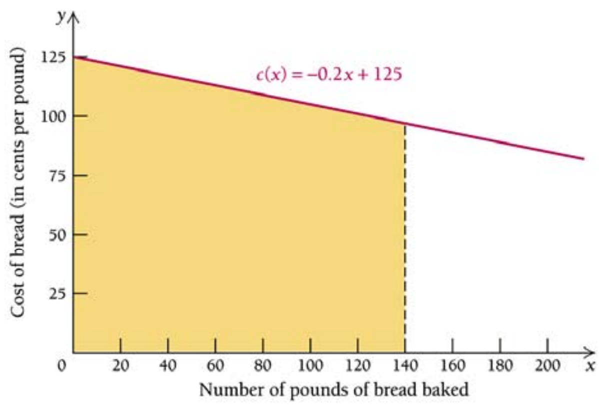
$$c(x) = -0.2x + 125, \quad x < 500$$

where x is the number of pounds of bread baked. Find the total cost of baking 140 lb of bread. Disregard fixed costs.

We will use the area under the graph of $c(x)$ to find the total cost of baking 140 lb of bread.

4.1 The Area Under a Graph

Example 1 (continued):



Note that the area is a trapezoid.

4.1 The Area Under a Graph

Example 1 (continued):

Total cost of baking

140 lb. bread = Area of Trapezoid

$$= \frac{1}{2}h\left(b_1 + b_2\right)$$

$$= \frac{1}{2}\cdot 140\left(c(0) + c(140)\right)$$

$$= \frac{1}{2}\cdot 140\left(125 + 97\right)$$

$$= 15{,}540¢ \quad \text{or} \quad \$155.40$$

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4.1 The Area Under a Graph

Example 1 (concluded):

Check by examining the units involved:

Cost per pound · Number of pounds

¢/lb · lb = ¢

Covington's total cost of baking 140 lb of sourdough bread is $155.40.

4.1 The Area Under a Graph

Example 2: Raggs, Ltd., determines that the marginal cost, in dollars, when suits are manufactured is

$$C'(x) = 0.0008x^2 - 0.45x + 100.$$

Ignoring fixed costs, approximate the total cost of producing 400 suits by using four rectangles.

4.1 The Area Under a Graph

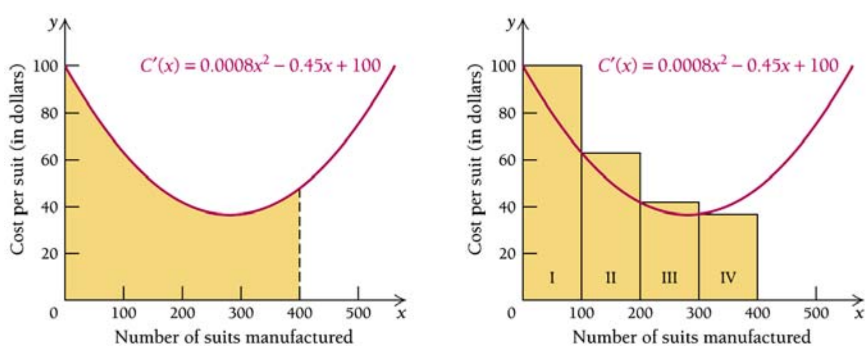
Example 2 (continued):

Solution

We divide the interval [0, 600] into four subintervals, each of length $\Delta x = 400/4 = 100$. To determine the height of each rectangle, we use the left endpoint of each subinterval.

4.1 The Area Under a Graph

Example 2 (continued):



4.1 The Area Under a Graph

Example 2 (concluded):

Then, we have the total cost of producing 400 suits is

$$\approx \text{Area I} + \text{Area II} + \text{Area III} + \text{Area IV}.$$

$$\approx C'(0)\cdot 100 + C'(100)\cdot 100$$
$$+C'(200)\cdot 100 + C'(300)\cdot 100$$
$$\approx \$100\cdot 100 + \$63\cdot 100 + \$42\cdot 100 + \$37\cdot 100$$
$$\approx \$10,000 + \$6300 + \$4200 + 3700 \approx \$24,200$$

Total cost of producing 400 suits is approximately \$24,200.

4.1 The Area Under a Graph

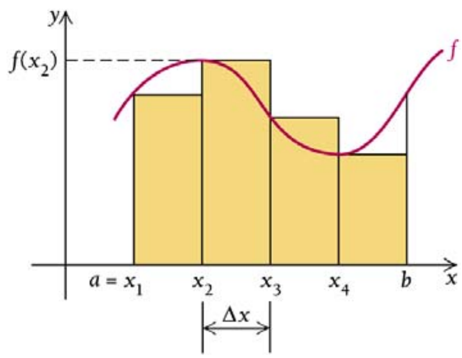
Riemann Sums:

In Example 2, we approximated the total cost of manufacturing 400 suits. How might we make that approximation more exact? We do so by using more rectangles that have a narrower width. This type of calculation is most easily accomplished with the use of summation notation.

4.1 The Area Under a Graph

Riemann Sums (continued):

In the following figure, $[a, b]$ is divided into four subintervals, each having width $\Delta x = (b - a)/4$.



The heights of the rectangles are $f(x_1)$, $f(x_2)$, $f(x_3)$ and $f(x_4)$.

4.1 The Area Under a Graph

Riemann Sums (concluded):

The area of the region under the curve is approximately the sum of the areas of the four rectangles:

$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x.$$

We can denote this sum with summation, or sigma, notation, which uses the Greek capital letter sigma, or Σ:

$$\sum_{i=1}^{4} f(x_i)\Delta x, \quad \text{or} \quad \sum\nolimits_{i=1}^{4} f(x_i)\Delta x.$$

This is read "the sum of the product $f(x_i)\Delta x$ from $i = 1$ to $i = 4$." To recover the original expression, we substitute the numbers 1 through 4 successively for i in $f(x_i)\Delta x$ and write plus signs between the results.

4.1 The Area Under a Graph

Example 3: Write summation notation for

$$2 + 4 + 6 + 8 + 10.$$

Note that we are adding consecutive values of 2.

$$2+4+6+8+10=\sum_{i=1}^{5}2i$$

4.1 The Area Under a Graph

Example 4: Write summation notation for:

$$g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_{19})\Delta x.$$

$$g(x_1)\Delta x + g(x_2)\Delta x + \cdots + g(x_{19})\Delta x = \sum_{i=1}^{19} g(x_i)\Delta x$$

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4.1 The Area Under a Graph

Example 5: Express $\sum_{i=1}^{4} 3^i$ without using summation notation.

$$\sum_{i=1}^{4} 3^i = 3^1 + 3^2 + 3^3 + 3^4 = 120$$

4.1 The Area Under a Graph

Example 6: Express $\sum_{i=1}^{30} h(x_i)\Delta x$ without using summation notation.

$$\sum_{i=1}^{30} h(x_i)\Delta x = h(x_1)\Delta x + h(x_2)\Delta x + \cdots + h(x_{30})\Delta x$$

4.1 The Area Under a Graph

Example 7: Consider the graph of:

$$f(x) = 600x - x^2$$

over the interval [0, 600].

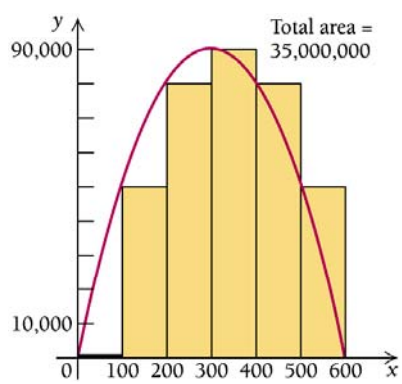
a) Approximate the area by dividing the interval into 6 subintervals.
b) Approximate the area by dividing the interval into 12 subintervals.

4.1 The Area Under a Graph

Example 7 (continued):
a) We divide [0, 600] into 6 intervals of size

$$\Delta x = \frac{600 - 0}{6} = 100,$$

with x_i ranging from $x_1 = 0$ to $x_6 = 500$.



4.1 The Area Under a Graph

Example 7 (continued):

Thus, the area under the curve is approximately

$$\sum_{i=1}^{6} f(x_i)\Delta x = f(0)\cdot 100 + f(100)\cdot 100 + f(200)\cdot 100$$
$$+ f(300)\cdot 100 + f(400)\cdot 100 + f(500)\cdot 100$$
$$= 0\cdot 100 + 50{,}000\cdot 100 + 80{,}000\cdot 100$$
$$+90{,}000\cdot 100 + 80{,}000\cdot 100 + 50{,}000\cdot 100$$
$$= 35{,}000{,}000$$

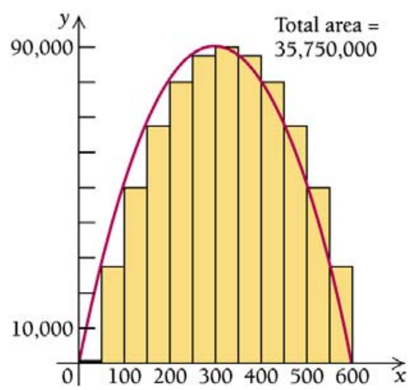
4.1 The Area Under a Graph

Example 7 (continued):

b) We divide [0, 600] into 12 intervals of size

$$\Delta x = \frac{600 - 0}{12} = 50,$$

with x_i ranging from $x_1 = 0$ to $x_{12} = 550$.



4.1 The Area Under a Graph

Example 7 (concluded):

Thus, the area under the curve is approximately

$$\sum_{i=1}^{12} f(x_i)\Delta x = f(0)\cdot 50 + f(50)\cdot 50 + f(100)\cdot 50 + f(150)\cdot 50$$

$$+ f(200)\cdot 50 + f(250)\cdot 50 + f(300)\cdot 50 + f(350)\cdot 50$$

$$+ f(400)\cdot 50 + f(450)\cdot 50 + f(500)\cdot 50 + f(550)\cdot 50$$

$$= 0\cdot 50 + 27,500\cdot 50 + 50,000\cdot 50 + 67,500\cdot 50$$

$$+80,000\cdot 50 + 87,500\cdot 50 + 90,000\cdot 50 + 87,500\cdot 50$$

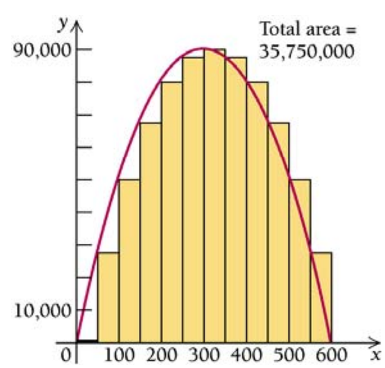
$$+80,500\cdot 50 + 67,500\cdot 50 + 50,000\cdot 50 + 27,500\cdot 50$$

$$= 35,750,000$$

4.1 The Area Under a Graph

Example 8: Approximate the area under the graph of $f(x) = 0.1x^3 - 2.3x^2 + 12x + 25$ over the interval [1, 16] by dividing the interval into 5 subintervals.

We divide [1,16] into 5 subintervals of size $\Delta x = (16 - 1)/5 = 3$, with x_i ranging from $x_1 = 1$ to $x_5 = 13$.



4.1 The Area Under a Graph

Example 8 (concluded):

The area under the curve from 1 to 16 is Approximately

$$\sum_{i=1}^{5} f(x_i)\Delta x = f(1)\cdot 3 + f(4)\cdot 3 + f(7)\cdot 3 + f(10)\cdot 3 + f(13)\cdot 3$$

$$= 34.8\cdot 3 + 42.6\cdot 3 + 30.6\cdot 3 + 15\cdot 3 + 12\cdot 3$$

$$= 405$$

Applications of Approximating Areas

EXAMPLE

The velocity of a car (in feet per second) is recorded from the speedometer every 10 seconds, beginning 5 seconds after the car starts to move. See Table 2. Use a Riemann sum to estimate the distance the car travels during the first 60 seconds. (Note: Each velocity is given at the middle of a 10-second interval. The first interval extends from 0 to 10, and so on.)

TABLE 2 A Car's Velocity

Time	5	15	25	35	45	55
Velocity	20	44	32	39	65	80

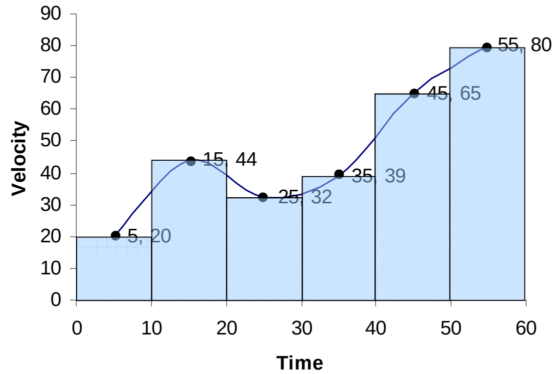
SOLUTION

Since measurements of the car's velocity were taken every ten seconds, we will use $\Delta x = 10$. *Now, upon seeing the graph of the car's velocity, we can construct a Riemann sum to estimate how far the car traveled.*

Applications of Approximating Areas

CONTINUED

Note: These rectangles were constructed using the midpoint of the intervals.



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$$v(5)\Delta t + v(15)\Delta t + v(25)\Delta t + v(35)\Delta t + v(45)\Delta t + v(55)\Delta t$$

$$= [v(5) + v(15) + v(25) + v(35) + v(45) + v(55)]\Delta t$$

$$= [20 + 44 + 32 + 39 + 65 + 80] \cdot 10$$

$$= 2800$$

Therefore, we estimate that the distance the car traveled is 2800 feet.