RATIONAL FUNCTIONS

Section 4.2: Properties of Rational Functions

Learning Objectives:

1. Find the Domain of a Rational Function (p. 192)
2. Find the Vertical Asymptotes of a Rational Function (p. 195)
3. Find the Horizontal or Oblique Asymptotes of a Rational Function (p. 196)

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

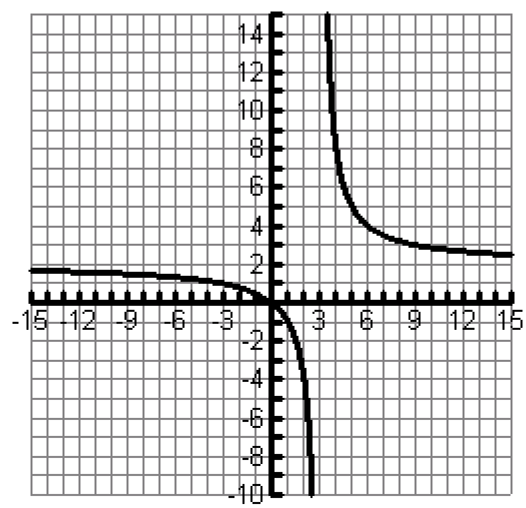
where p and q are polynomial functions and q is not the zero polynomial.

The domain is the set of all real numbers except those for which the denominator q is 0.

Vertical and Horizontal Asymptotes of Rational Functions $f(x) = \frac{p(x)}{q(x)}$

Example 1: Consider the rational function $f(x) = \frac{2x}{x-3}$

What is the domain of *f*?



- The line $x = a$ is a ***vertical asymptote*** of $f(x)$ if $f(x) \to \pm\infty$ as $x \to a^-$ or $x \to a^+$. To find the *vertical asymptote(s)* algebraically set $q(x) = 0$ and find the real zeros.

x	*f(x)*	*x*	*f(x)*
3.2		2.8	
3.1		2.89	
3.01		2.9	
3.001		2.98	
3.0001		2.99	
3.00001		2.999	

What is the vertical asymptote of f?

- The line $y = b$ is a ***horizontal asymptote*** of $f(x)$ if $f(x) \to b$ as $x \to \pm\infty$.

x	*f(x)*	*x*	*f(x)*
10		-10	
100		-100	
1000		-1000	
10000		-10000	

What is the horizontal asymptote of f?

Consider the rational function $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0}{b_n x^n + \ldots + b_2 x^2 + b_1 x + b_0}$.

If the degree of *p(x)* equals the degree of *q(x)* then the horizontal asymptote is $y = \frac{a}{b}$.

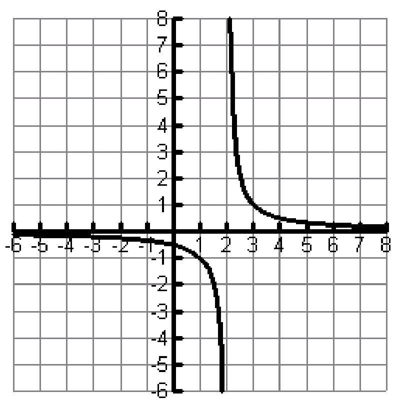
If the degree of *p(x)* is less than the degree of *q(x)* then the horizontal asymptote is $y = 0$.

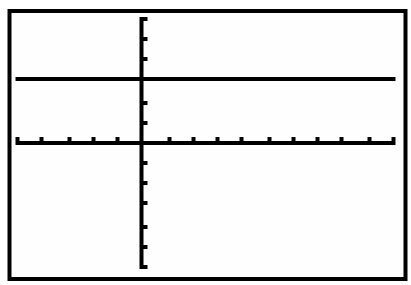
If the degree of *p(x)* is greater than the degree of *q(x)* then there is no horizontal asymptote.

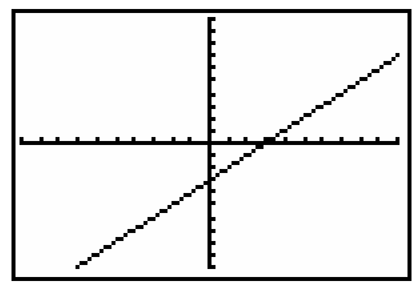
Holes:

State the domain of each function.

$g(x) = \frac{x+2}{x^2-4} =$ $\qquad h(x) = \frac{3x+9}{x+3}$ $\qquad k(x) = \frac{x^2-9}{x+3}$







X	Y1
-3	-.2
-2	ERROR
-1	-.3333
0	-.5
1	-1
2	ERROR
3	1

X=3

X	Y1
-6	3
-5	3
-4	3
-3	ERROR
-2	3
-1	3
0	3

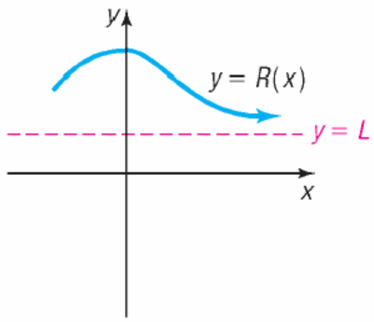
X= -6

X	Y1
-6	-9
-5	-8
-4	-7
-3	ERROR
-2	-5
-1	-4
0	-3

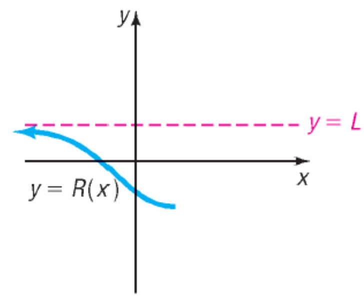
X= -6

Give the coordinates of any holes.

Let R denote a function:
If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.
If, as x approaches some number c, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of R. The graph of R never intersects a vertical asymptote.



(a) End behavior: As $x \to \infty$, the values of $R(x)$ approach L [$\lim_{x \to \infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.



(b) End behavior: As $x \to -\infty$, the values of $R(x)$ approach L [$\lim_{x \to -\infty} R(x) = L$]. That is, the points on the graph of R are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

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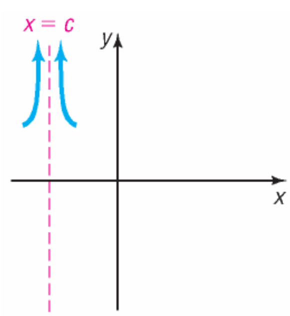
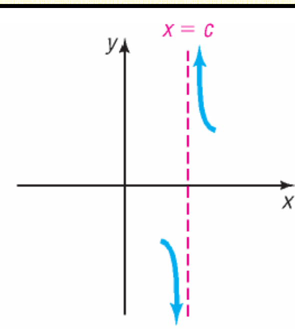


(c) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x \to c^-} R(x) = \infty$; $\lim_{x \to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.



(d) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x \to c^-} R(x) = -\infty$; $\lim_{x \to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

Let R denote a function:
If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y = L$ is a **horizontal asymptote** of the graph of R.
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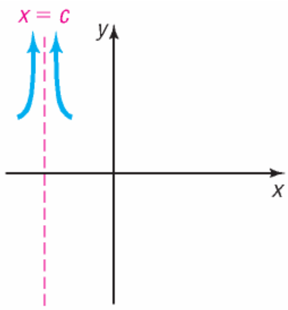

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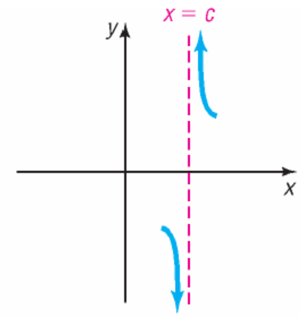

(d) As x approaches c, the values of $|R(x)| \to \infty$ [$\lim_{x \to c^-} R(x) = -\infty$; $\lim_{x \to c^+} R(x) = \infty$]. That is, the points on the graph of R are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

SUMMARY Finding Horizontal and Oblique Asymptotes of a Rational Function R

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is n and the degree of the denominator is m.

1. If $n < m$ (the degree of the numerator is less than the degree of the denominator), then R is a proper rational function, and the graph of R will have the horizontal asymptote $y = 0$ (the x-axis).

2. If $n \geq m$ (the degree of the numerator is greater than or equal to the degree of the denominator), then R is improper. Here long division is used.

(a) If $n = m$ (the degree of the numerator equals the degree of the denominator), the quotient obtained will be the number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.

(b) If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.

(c) If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), the quotient obtained is a polynomial of degree 2 or higher, and R has neither a horizontal nor an oblique asymptote. In this case, for $|x|$ unbounded, the graph of R will behave like the graph of the quotient.

NOTE The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote. ■

EXAMPLE How to Analyze the Graph of a Rational Function

Analyze the graph of the rational function: $R(x) = \dfrac{x^2 - 4}{x^2 + 3x - 4}$

Step-by-Step Solution

STEP 1 Factor the numerator and denominator of *R*. Find the domain of the rational function.

STEP 2 Write *R* in lowest terms.

STEP 3 Locate the intercepts of the graph.

STEP 4 Test for symmetry. If $R(-x) = R(x)$, the function is even and its graph will be symmetric with respect to the *y*-axis. If $R(-x) = -R(x)$, the function is odd and its graph will be symmetric with respect to the origin.

STEP 5 Locate the vertical asymptotes.

STEP 6 Locate the horizontal or oblique asymptotes. Determine points, if any, at which the graph of *R* intersects these asymptotes.

STEP 7 Graph *R* using a graphing utility.

STEP 8 Use the results obtained in Steps 1 through 7 to graph *R* by hand.

Analyzing the Graph of a Rational Function

STEP 1: Factor the numerator and denominator of *R*. Find the domain of the rational function.

STEP 2: Write *R* in lowest terms.

STEP 3: Locate the intercepts of the graph. The *x*-intercepts, if any, of $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms satisfy the equation $p(x) = 0$. The *y*-intercept, if there is one, is $R(0)$.

STEP 4: Test for symmetry. Replace x by $-x$ in $R(x)$. If $R(-x) = R(x)$, there is symmetry with respect to the *y*-axis; if $R(-x) = -R(x)$, there is symmetry with respect to the origin.

STEP 5: Locate the vertical asymptotes. The vertical asymptotes, if any, of $R(x) = \dfrac{p(x)}{q(x)}$ in lowest terms are found by identifying the real zeros of $q(x)$. Each zero of the denominator gives rise to a vertical asymptote.

STEP 6: Locate the horizontal or oblique asymptotes, if any, using the procedure given in Section 4.2. Determine points, if any, at which the graph of *R* intersects these asymptotes.

STEP 7: Graph *R* using a graphing utility.

STEP 8: Use the results obtained in Steps 1 through 7 to graph *R* by hand.

EXAMPLE Finding the Least Cost of a Can

Reynolds Metal Company manufactures aluminum cans in the shape of a cylinder with a capacity of 500 cubic centimeters $\left(\frac{1}{2}\text{ liter}\right)$. The top and bottom of the can are made of a special aluminum alloy that costs 0.05¢ per square centimeter. The sides of the can are made of material that costs 0.02¢ per square centimeter.

(a) Express the cost of material for the can as a function of the radius r of the can.

(b) Use a graphing utility to graph the function $C = C(r)$.

(c) What value of r will result in the least cost?

(d) What is this least cost?