Scott

# RATIONAL FUNCTIONS

## Section 4.2: Properties of Rational Functions

**Learning Objectives:**

1. Find the Domain of a Rational Function (p. 192)
2. Find the Vertical Asymptotes of a Rational Function (p. 195)
3. Find the Horizontal or Oblique Asymptotes of a Rational Function (p. 196)

A **rational function** is a function of the form

$$R(x) = \frac{p(x)}{q(x)}$$

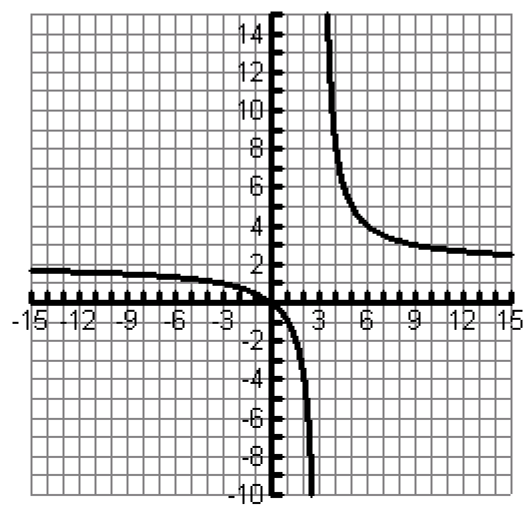
where $p$ and $q$ are polynomial functions and $q$ is not the zero polynomial.

The domain is the set of all real numbers except those for which the denominator $q$ is 0.

**Vertical and Horizontal Asymptotes of Rational Functions** $f(x) = \frac{p(x)}{q(x)}$

Example 1: Consider the rational function $f(x) = \frac{2x}{x-3}$

What is the domain of *f*?




- The line $x = a$ is a ***vertical asymptote*** of *f(x)* if $f(x) \to \pm\infty$ as $x \to a^-$ or $x \to a^+$. To find the *vertical asymptote(s)* algebraically set $q(x) = 0$ and find the real zeros.

| *x* | *f(x)* | *x* | *f(x)* |
|---|---|---|---|
| 3.2 | | 2.8 | |
| 3.1 | | 2.89 | |
| 3.01 | | 2.9 | |
| 3.001 | | 2.98 | |
| 3.0001 | | 2.99 | |
| 3.00001 | | 2.999 | |
| | | | |

*What is the vertical asymptote of f?*

- The line $y = b$ is a ***horizontal asymptote*** of *f(x)* if $f(x) \to b$ as $x \to \pm\infty$.

| *x* | *f(x)* | *x* | *f(x)* |
|---|---|---|---|
| 10 | | -10 | |
| 100 | | -100 | |
| 1000 | | -1000 | |
| 10000 | | -10000 | |
| | | | |

*What is the horizontal asymptote of f?*

Consider the rational function $f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0}{b_n x^n + \ldots + b_2 x^2 + b_1 x + b_0}$.

If the degree of *p(x)* equals the degree of *q(x)* then the horizontal asymptote is $y = \frac{a}{b}$.

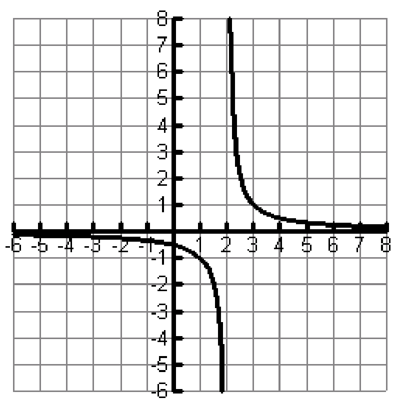
If the degree of *p(x)* is less than the degree of *q(x)* then the horizontal asymptote is $y = 0$.

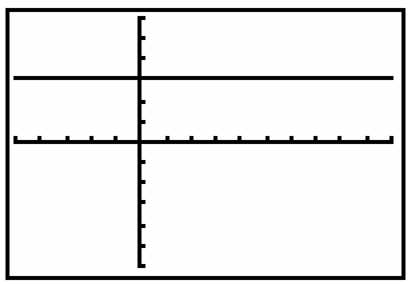
If the degree of *p(x)* is greater than the degree of *q(x)* then there is no horizontal asymptote.

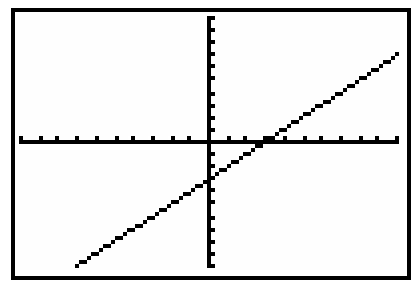
**Holes:**

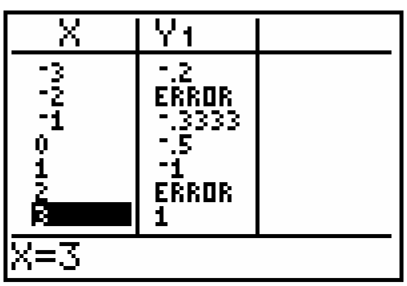
State the domain of each function.

$g(x) = \frac{x+2}{x^2-4} =$ $\qquad h(x) = \frac{3x+9}{x+3}$ $\qquad k(x) = \frac{x^2-9}{x+3}$



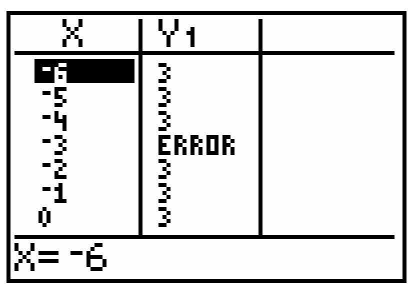







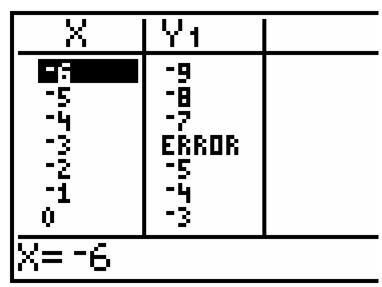
| X | Y1 |
|---|---|
| -3 | -.2 |
| -2 | ERROR |
| -1 | -.3333 |
| 0 | -.5 |
| 1 | -1 |
| 2 | ERROR |
| 3 | 1 |

X=3



| X | Y1 |
|---|---|
| -6 | 3 |
| -5 | 3 |
| -4 | 3 |
| -3 | ERROR |
| -2 | 3 |
| -1 | 3 |
| 0 | 3 |

X= -6

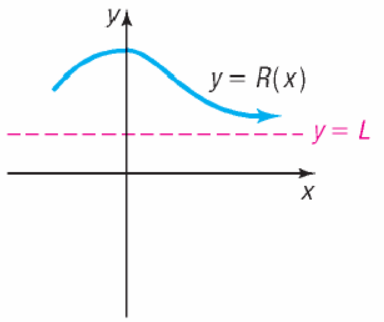


| X | Y1 |
|---|---|
| -6 | -9 |
| -5 | -8 |
| -4 | -7 |
| -3 | ERROR |
| -2 | -5 |
| -1 | -4 |
| 0 | -3 |

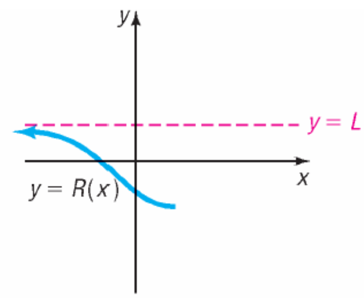
X= -6

Give the coordinates of any holes.

Let $R$ denote a function:
If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number $L$, then the line $y = L$ is a **horizontal asymptote** of the graph of $R$.
If, as $x$ approaches some number $c$, the values $|R(x)| \to \infty$, then the line $x = c$ is a **vertical asymptote** of the graph of $R$. The graph of $R$ never intersects a vertical asymptote.




**(a)** End behavior: As $x \to \infty$, the values of $R(x)$ approach $L$ $[\lim_{x\to\infty} R(x) = L]$. That is, the points on the graph of $R$ are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.




**(b)** End behavior: As $x \to -\infty$, the values of $R(x)$ approach $L$ $[\lim_{x\to-\infty} R(x) = L]$. That is, the points on the graph of $R$ are getting closer to the line $y = L$; $y = L$ is a horizontal asymptote.

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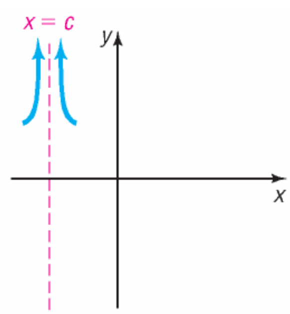
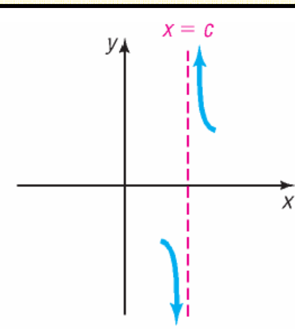



**(c)** As $x$ approaches $c$, the values of $|R(x)| \to \infty$ $[\lim_{x\to c^-} R(x) = \infty;$ $\lim_{x\to c^+} R(x) = \infty]$. That is, the points on the graph of $R$ are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.




**(d)** As $x$ approaches $c$, the values of $|R(x)| \to \infty$ $[\lim_{x\to c^-} R(x) = -\infty;$ $\lim_{x\to c^+} R(x) = \infty]$. That is, the points on the graph of $R$ are getting closer to the line $x = c$; $x = c$ is a vertical asymptote.

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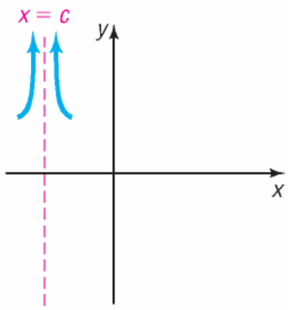


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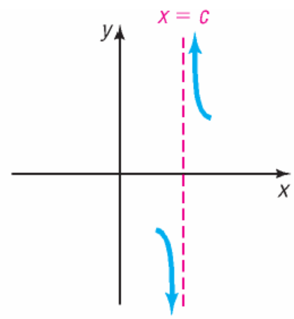


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## SUMMARY Finding Horizontal and Oblique Asymptotes of a Rational Function $R$

Consider the rational function

$$R(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

in which the degree of the numerator is $n$ and the degree of the denominator is $m$.

**1.** If $n < m$ (the degree of the numerator is less than the degree of the denominator), then $R$ is a proper rational function, and the graph of $R$ will have the horizontal asymptote $y = 0$ (the $x$-axis).

**2.** If $n \geq m$ (the degree of the numerator is greater than or equal to the degree of the denominator), then $R$ is improper. Here long division is used.

(a) If $n = m$ (the degree of the numerator equals the degree of the denominator), the quotient obtained will be the number $\frac{a_n}{b_m}$, and the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote.

(b) If $n = m + 1$ (the degree of the numerator is one more than the degree of the denominator), the quotient obtained is of the form $ax + b$ (a polynomial of degree 1), and the line $y = ax + b$ is an oblique asymptote.

(c) If $n \geq m + 2$ (the degree of the numerator is two or more greater than the degree of the denominator), the quotient obtained is a polynomial of degree 2 or higher, and $R$ has neither a horizontal nor an oblique asymptote. In this case, for $|x|$ unbounded, the graph of $R$ will behave like the graph of the quotient.

NOTE The graph of a rational function either has one horizontal or one oblique asymptote or else has no horizontal and no oblique asymptote. ■